

COMPLETE INTERSECTION POINTS ON GENERAL SURFACES IN \mathbb{P}^3

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ABSTRACT. In this paper we consider the existence of complete intersection points of type (a, b, c) , on the generic degree d surface of \mathbb{P}^3 . For any choice of a, b, c we resolve the existence question asymptotically, i.e. for all $d \gg 0$. For small values of a, b, c we resolve the existence problem completely.

1. INTRODUCTION

A recurrent theme in classical projective geometry is the study of special subvarieties of some given family of varieties, e.g. how many isolated singular points can a surface of degree d in \mathbb{P}^3 have? when is it true that the members of a certain family of varieties contain rational curves? contain a linear space of some positive dimension? Other examples of similar questions can be easily provided by the reader.

The study of the special case of complete intersection subvarieties of hypersurfaces in \mathbb{P}^n has been the subject of a great deal of research. It was known to Severi [Sev06] that for $n \geq 4$ the only complete intersections, of codimension one, on a general hypersurface are obtained by intersecting that hypersurface with another.

This observation was extended to \mathbb{P}^3 by Noether (and Lefschetz) [Lef21, GH85] for general hypersurfaces of degree ≥ 4 . These ideas were further generalized by Grothendieck [Gro05].

In [CCG08], we proposed a new approach to the problem of studying complete intersection subvarieties of hypersurfaces. This approach used a mix of projective geometry and commutative algebra and is more elementary and direct than, for example, the approach of Grothendieck. With our approach we were able to give a complete description of the situation for complete intersections of codimension r in \mathbb{P}^n which lie on a general hypersurface of degree d whenever $2r \leq n + 2$. The main result of [CCG08] is the following:

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a generic degree d hypersurface, with $n, d > 1$. Then X contains a complete intersection of type (a_1, \dots, a_r) ,*

with $2r \leq n + 2$, and the a_i all less than d , in the following (and only in the following) instances:

- $n = 2$: then $r = 2$, d arbitrary and a_1 and a_2 can assume any value less than d ;
- $n = 3, r = 2$: for $d \leq 3$ we have that a_1 and a_2 can assume any value less than d ;
- $n = 4, r = 3$: for $d \leq 5$ we have that a_1, a_2 and a_3 can assume any value less than d ;
- $n = 6, r = 4$ or $n = 8, r = 5$: for $d \leq 3$ we have that a_1, \dots, a_r can assume any value less than d ;
- $n = 5, 7$ or $n > 8$, $2r = n + 1$ or $2r = n + 2$: we have only linear spaces on quadrics, i.e. $d = 2$ and $a_1 = \dots = a_r = 1$.

In this paper, we are interested in the first case not covered by Theorem 1.1. Namely, the case $n = 3, r = 3$, i.e. complete intersection points on surfaces of \mathbb{P}^3 . Although this a very natural question, we are not aware of any reference to the subject in the literature. Using the methods of [CCG08] we prove the following:

Theorem 1.2. *For non-negative integers a, b, c, d , such that $a \leq b \leq c < d$ we have the following:*

- if $a \leq 4$, then the generic degree d surface of \mathbb{P}^3 contains a $CI(a, b, c)$;
- if $a = 5, b \leq 11$, then the generic degree d surface of \mathbb{P}^3 contains a $CI(5, b, c)$; if $a = 5, b = 12$ and $c = 12$ then the generic degree d surface of \mathbb{P}^3 contains a $CI(5, 12, 12)$; if $a = 5, b = 12$ and $c \geq 13$ then the generic degree $d \geq 2c + 15$ surface does not contain a $CI(5, 12, c)$; if $a = 5$ and $b \geq 13$, then the generic degree $d \geq b + c + 2$ surface does not contain a $CI(5, b, c)$;
- if $a = 6, b \leq 7$, then the generic degree d surface of \mathbb{P}^3 contains a $CI(6, b, c)$; if $a = 6, b = 8$ and $c = 8, 9$ then the generic degree d surface of \mathbb{P}^3 contains a $CI(6, 8, c)$; if $a = 6, b = 8$ and $c \geq 10$ then the generic degree $d \geq 2c + 12$ surface does not contain a $CI(6, 8, c)$; if $a = 6$ and $b \geq 9$, then the generic degree $d \geq b + c + 3$ surface does not contain a $CI(6, b, c)$;
- if $a \geq 7$, then the generic degree $d \geq a + b + c - 3$ surface of \mathbb{P}^3 does not contain a $CI(a, b, c)$.

Notice that Theorem 1.2 gives a complete asymptotic solution to the existence problem for $CI(a, b, c)$ on a general surface of degree d in \mathbb{P}^3 . More precisely,

Corollary 1.3. *Let $a \leq b \leq c < d$ be integers. Then for $d \gg 0$ the generic degree d surface contains a $CI(a, b, c)$ if:*

- $a \leq 4$;
- $a = 5, b \leq 11$;
- $a = 5, b = 12, c = 12$;
- $a = 6, b \leq 7$;
- $a = 6, b = 8, c = 8, 9$.

and does not contain a $CI(a, b, c)$ in all other cases.

Remark 1.4. We also notice that the kind of asymptotic problem we solved above can only be considered for points. More precisely, if we choose a family \mathcal{F} of subschemes of \mathbb{P}^n we can ask the following: is it true that for $d \gg 0$ the generic degree d hypersurface of \mathbb{P}^n contains an object of the family \mathcal{F} ?

Using a standard incidence correspondence argument, it is easy to see that a positive answer can be given only if

$$\dim \mathcal{F} + 1 - h_{\mathcal{F}}(d) \geq 0,$$

where $h_{\mathcal{F}}(d)$ is the Hilbert polynomial of the objects in \mathcal{F} . Clearly this can be the case only if $h_{\mathcal{F}}(d)$ is bounded and hence constant. This implies that \mathcal{F} is a family parameterizing 0-dimensional schemes.

The paper is structured as follows: in Section 2, we formalize the question we want to study and we treat the first simple instances; in Section 3, we recall the results we need from [CCG08]; in Sections 4, 5 and 6 we apply our method to produce the intermediate results necessary to prove Theorem 1.2. Finally, in Section 7, we prove Theorem 1.2 and we state a conjecture for the expected behavior in the cases which still remain open.

In the proof of Theorem 6.3 we used the computer algebra system CoCoA [CoC04] for which we thank the developers of the software.

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2. THE QUESTION

In this paper we study complete intersection points in projective three space. We say that $\mathbb{X} \subset \mathbb{P}^3$ is a complete intersection 0-dimensional scheme if its ideal $I_{\mathbb{X}} = (F, G, H)$ where the forms F, G and H are a regular sequence in the ring $R = \mathbb{C}[x_0, \dots, x_3]$. Moreover, if $\deg F = a, \deg G = b$ and $\deg H = c$ we say that \mathbb{X} is a *complete intersection*

of type (a, b, c) . We will always assume $a \leq b \leq c$ and we will write $CI(a, b, c)$ to describe a complete intersection of type (a, b, c) .

Our basic question is: *for which integers a, b, c and d does the general degree d surface of \mathbb{P}^3 contain a $CI(a, b, c)$?*

There are cases where the answer is straightforward. If $d = c$, the answer is clearly affirmative as we are cutting a complete intersection curve of type (a, b) with a surface of degree d (similarly for $d = a$ or $d = b$). If $d < a$, the answer is negative as no form of degree less than a belongs to the ideal of a $CI(a, b, c)$, and similarly for $a < d < b$ as a generic form is irreducible. If $b < d < c$, then we are really looking for a complete intersection of type (a, b) on the generic degree d surface, and this is dealt with in Theorem 1.1. Hence, it is enough to focus on the following refinement of our question: *for which integers a, b, c and d , $a \leq b \leq c < d$, does the general degree d surface of \mathbb{P}^3 contain a $CI(a, b, c)$?*

3. TECHNICAL FACTS

We will treat this question using the method introduced in [CCG08]. Our method proceeds as follows: translate the problem of finding a $CI(a, b, c)$ on a general surface of degree d , say $M = 0$, as the problem of writing M as

$$M = FF' + GG' + HH'$$

where F, G, H and F', G', H' are forms of degree a, b, c and $d-a, d-b, d-c$ respectively. As M is generic, this decomposition problem is actually a problem about joins of varieties of splitting forms. Then we use Terracini's lemma to translate the computation of the dimension of the join, into a Hilbert function computation. Namely, as first observed in [Mam54], the tangent space to the variety of splitting forms at the point $[FF']$ corresponds to the degree d homogeneous piece of the ideal (F, F') . Thus, the tangent space at M to the join corresponds to the degree d homogeneous piece of the ideal spanned by F, F', G, G', H, H' . For more details we refer the reader to [CCG08].

In particular we will need the following (see [CCG08, Lemma 4.3]):

Lemma 3.1. *For given integers a, b, c and d , such that $a \leq b \leq c < d$, the following are equivalent facts:*

- (1) *The general degree d surface of \mathbb{P}^3 contains a $CI(a, b, c)$;*
- (2) *For a generic choice of forms $F, G, H, H', G', F' \in R$ of degrees $a, b, c, d-c, d-b, d-a$ one has that*

$$H \left(\frac{R}{(F, G, H, H', G', F')}, d \right) = 0$$

where $H(\cdot, d)$ denotes the Hilbert function in degree d .

Using Lemma 3.1 we translate our geometric question into a purely algebraic one. In particular, we can take advantage of results about the Lefschetz property [Sta80, Ani86] to deal with our question.

As F, G, H and H' are a regular sequence in R we have a good understanding of the ring

$$W = \frac{R}{(F, G, H, H')}$$

and we will use this to study the Hilbert function of the ring

$$\frac{R}{(F, G, H, H', G', F')} \simeq \frac{W}{([F'], [G'])},$$

where $[\cdot]$ denotes the class in W .

Via the Koszul complex we compute the minimal free resolution of W :

$$(1) \quad 0 \leftarrow W \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow M_4 \leftarrow 0$$

where

$$M_0 = R,$$

$$M_1 = R(-a) \oplus R(-b) \oplus R(-c) \oplus R(-d+c),$$

$$M_2 = R(-a-b) \oplus R(-a-c) \oplus R(-a-d+c) \oplus R(-b-c) \oplus R(-b-d+c) \oplus R(-d),$$

$$M_3 = R(-a-b-c) \oplus R(-a-b-d+c) \oplus R(-a-d) \oplus R(-b-d)$$

$$M_4 = R(-a-b-d).$$

We also notice that (see [CCG08, Lemma 4.1 and Remark 4.2]):

Lemma 3.2. *The following are equivalent:*

- for integers $a \leq b \leq c < d$ a $CI(a, b, c)$ exists on the generic degree d surface of \mathbb{P}^3 ;
- for integers $a' \leq b' \leq c' \leq d$ a $CI(a', b', c')$ exists on the generic degree d surface of \mathbb{P}^3 , where $a = a'$ or $a + a' = d$, and $b = b'$ or $b + b' = d$, and $c = c'$ or $c + c' = d$.

Remark 3.3. Using Lemma 3.2 we can study our question for integers $a \leq b \leq c < d/2$ and produce a complete answer for the general case. In fact, either $a \leq d/2$ or $a' \leq d/2$.

4. THE $a \leq 4$ CASE

Here we use Stanley's result [Sta80] showing that the quotient of $R = \mathbb{C}[x_0, \dots, x_3]$ by four generic forms has the Strong Lefschetz Property. More precisely, given generic forms $F, G, H, F', G' \in R$, of degrees $a, b, c, d - c, d - b$, we consider $W = R/(F, G, H, H')$. Then the multiplication by the class of G' has maximal rank. Hence the sequence

$$W(-d + b) \rightarrow W \rightarrow \frac{W}{([G'])} \rightarrow 0$$

produces $H(W/([G']), d) = \max\{H(W, d) - H(W, b), 0\}$.

Proposition 4.1. *For any choice of a, b, c and d positive integers such that $a \leq 4 \leq b \leq c$ and $d \geq a + b + c - 3$, the general degree d surface in \mathbb{P}^3 contains a $CI(a, b, c)$.*

Proof. Using Lemma 3.3 it is enough to consider the case when $a \leq b \leq c \leq \frac{d}{2}$. Using Proposition 3.1 part (2) we only have to show that

$$H(W/([G']), d) = \max\{H(W, d) - H(W, b), 0\} = 0.$$

By the resolution of W given in (1) we immediately get:

- if $b < c$, then

$$\begin{aligned} H(W, b) &= \binom{b+3}{3} - \binom{b-a+3}{3} - 1 \\ &= 1/6a^3 - 1/2a^2b + 1/2ab^2 - a^2 + 2ab + 11/6a - 1; \end{aligned}$$

- if $b = c$, then

$$\begin{aligned} H(W, b) &= \binom{b+3}{3} - \binom{b-a+3}{3} - 2 \\ &= 1/6a^3 - 1/2a^2b + 1/2ab^2 - a^2 + 2ab + 11/6a - 2. \end{aligned}$$

Led by the resolution of W , we also consider the following polynomial

$$\begin{aligned} h(W, d) &= \\ &= \binom{d+3}{3} - \left[\binom{d-a+3}{3} + \binom{d-b+3}{3} + \binom{d-c+3}{3} + \binom{c+3}{3} \right] + \\ &+ \binom{d-a-b+3}{3} + \binom{d-a-c+3}{3} + \binom{c-a+3}{3} + \binom{d-b-c+3}{3} + \\ &+ \binom{c-b+3}{3} + 1 - \left[\binom{d-a-b-c+3}{3} + \binom{c-a-b+3}{3} \right], \end{aligned}$$

where $\binom{x}{3}$ is the polynomial $\frac{1}{6}x(x-1)(x-2)$. Making the computation we get

$$h(W, d) = 1/2a^2b + 1/2ab^2 - 2ab + 1.$$

Notice that, for given a, b and c such that $c - a - b \geq -3$ and $d \geq a + b + c - 3$, the evaluation of $h(W, d)$ coincides with the Hilbert function of W in degree d , i.e. $h(W, d) = H(W, d)$. Moreover, the inequalities

$$a \leq 4 \text{ and } c - a - b \leq -4$$

only hold when $a = 4$ and $b = c$ (recall that $a \leq b \leq c$) and in this case

$$H(W, d) = h(W, d) + \binom{c - a - b + 3}{3} = h(W, d) - 1.$$

Finally we compute $H(W, d) - H(W, b)$ distinguishing two cases.

The $a < 4$ or $b < c$ case. If $b < c$ we use the value of $H(W, b)$ and the equality $H(W, d) = h(W, d)$ previously determined to get

$$H(W, d) - H(W, b) = -1/6a^3 + a^2b + a^2 - 4ab - 11/6a + 2.$$

This polynomial is linear in b and it does not involve d and it is easy to see that for $a \leq 4$

$$H(W, d) - H(W, b) \leq 0.$$

When $a < 4$ and $b = c$ a completely analogous argument can be applied.

The $a = 4$ and $b = c$ case. Mutatis mutandis, we compute again and we get

$$H(W, d) - H(W, b) = -1/6a^3 + a^2b + a^2 - 4ab - 11/6a + 2,$$

hence the same polynomial of the previous case and this finishes the proof. \square

Proposition 4.1 gives an asymptotic result yielding that, when one of the degree of the CI is at most 4, then for d big enough a complete intersection of the given type exists on a generic surface of degree d . With a slightly more careful analysis this can be improved and the condition on d can be dropped.

Theorem 4.2. *Let a, b, c and d be integers such that $a \leq b \leq c < d$. If $a \leq 4$, then a $CI(a, b, c)$ exists on the generic degree d surface in \mathbb{P}^3 .*

Proof. Using Proposition 4.1 we have only to check values of d in the range $c < d \leq a + b + c - 4$. The idea is to use Lemma 3.2 to reduce the degree of the complete intersection not changing d so that Proposition 4.1 can be applied.

For $a = 2$, we consider the existence of a $CI(2, b, c)$ on the generic degree d surface for $c < d \leq b + c - 2$. Such a complete intersection

exists if the same happen for a $CI(2, d - c, d - b)$. But, by Proposition 4.1, this is the case as soon as

$$d \geq 2 + (d - c) + (d - b) - 3$$

and this equivalent to $d \leq b + c - 1$ which is actually the case. Similarly for $a = 3$.

The case $a = 4$ is treated in analogy with the previous ones, except for $d = b + c$. In this situation, applying Lemma 3.2, we have to study $CI(4, b, b)$'s on a generic surface of degree $d \geq b$. Repeating the same argument above we have to treat values of d in the range $b \leq d \leq 2b$. Proposition 6.1 gives the existence for all d , but for $d = 2b$. By Proposition 3.1 we have to consider the coordinate ring W of a complete intersection of type (b, b, b, b) and its Hilbert function $H(\cdot)$. Using the fact that multiplication by one form has maximal rank in W , we have only to compare $H(2b)$ and $H(2b - 4)$, but these values are the same being W a Gorenstein ring with socle degree $4b - 4$, and this finishes the proof. \square

5. THE CASE $a > 4$: NON-EXISTENCE RESULTS

In this section we will prove asymptotic non-existence results when $a > 4$. For non-negative integers a, b, c and d , $a \leq b \leq c < d$, we consider generic forms $F, G, H, H', G', F' \in R$ of degrees $a, b, c, d - c, d - b$ and $d - a$. Consider the ring $W = R/(F, G, H, H')$ and notice that, by a straightforward dimensional argument, if

$$H(W, a) + H(W, b) - H(W, d) < 0$$

then

$$H\left(\frac{W}{([G'], [F'])}, d\right) \neq 0.$$

Hence, by Lemma 3.1 (2), if $H(W, d) - H(W, a) - H(W, b) < 0$, then the generic degree d surface of \mathbb{P}^3 does not contain a $CI(a, b, c)$. Using this idea we prove the following:

Theorem 5.1. *Let $a \leq b \leq c$ and d be non-negative integers such that*

$$a = 5 \text{ and } b \geq 13$$

or

$$a = 6 \text{ and } b \geq 9$$

or

$$a \geq 7.$$

Then, for $d \geq a + b + c - 3$ the generic degree d surface of \mathbb{P}^3 does not contain a $CI(a, b, c)$.

In order to prove this theorem we need the following technical fact:

Lemma 5.2. *Let a, b, c be non-negative integers, such that $4 < a$. Assume that, for integers c_0 and d such that*

$$c_0 \geq b \text{ and } d > a + b + c_0 - 4,$$

one has the Hilbert function inequality

$$H(W, a) + H(W, b) - H(W, d) < 0,$$

where W is the ring

$$W = \frac{R}{(F, G, H, H')}$$

and the forms F, G, H and H' are generic and have degrees a, b, c_0 and $d - c_0$.

Then, if A, B, C and D are forms of degrees $a, b, c \geq c_0$ and $d - c$ and

$$W' = \frac{R}{(A, B, C, D)}$$

then the following inequality holds:

$$H(W', a) + H(W', b) - H(W', d) < 0,$$

for $d > a + b + c - 4$.

Proof. The key observation is that

$$H(W, d) = H(W, a + b - 4).$$

In fact, being W a Gorenstein ring, its Hilbert function is symmetric and $H(W, x) = H(W, y)$ if $x + y = d + a + b - 4$. Then, we compute

$$H(W, a) + H(W, b) - H(W, a + b - 4)$$

using the formulae in the proof of Proposition 4.1, for which we need the assumption on d . One sees that the final expression does not involve neither c or d and the proof follows. For example, in the case $a < b$ one gets $H(W, a) + H(W, b) - H(W, a + b - 4) < 0$ if and only if

$$b \geq \frac{\frac{1}{2}a^3 - a^2 + \frac{11}{2}a - 4}{(a^2 - 4a)}.$$

□

We can now prove Theorem 5.1.

Proof of Theorem 5.1. We let $b = c$ and we show that for the required values of a and b we have the inequality $H(W, a) + H(W, b) - H(W, d) < 0$. Then we apply Lemma 5.2 to get the result when $c \geq b$.

Again, we notice that $H(W, d) = H(W, a + b - 4)$.

We divide the proof in two cases depending on whether $a = b$ or $a < b$.

Case $a < b$.

Using the resolution of the ring W , the inequality

$$H(W, a) + H(W, b) - H(W, a + b - 4) \geq 0$$

is readily seen to be equivalent to

$$b \leq \frac{\frac{1}{2}a^3 - a^2 + \frac{11}{2}a - 4}{(a^2 - 4a)}.$$

Recalling that $a < b$ we get

$$H(W, a) + H(W, b) - H(W, d) \geq 0$$

only if

$$-\frac{1}{2}a^3 + 3a^2 + \frac{11}{2}a - 4 \geq 0$$

and this inequality holds if and only if

$$a = 5 \text{ or } a = 6.$$

Hence

$$H(W, a) + H(W, b) - H(W, d) \geq 0$$

implies

$$a = 5, b \leq 12$$

or

$$a = 6, b \leq 8.$$

Case $a = b$.

Computing we get

$$H(W, a) + H(W, b) - H(W, d) = -\frac{2}{3}a^3 + 4a^2 + \frac{11}{3}a - 3 \geq 0$$

only if $a < 7$ and this finishes the proof. \square

To prove some more non-existence results, we need the following:

Proposition 5.3. *Let $a \leq b \leq c < d$ and $d > 2c + b + a - 3$. If no $CI(a, b, c)$ exists on the generic degree d hypersurface, then it does not exist on the generic hypersurface of degree $d' > d$ either.*

Proof. It is enough to treat the case $d' = d + 1$. For generic forms F, G, H of degrees a, b and c let

$$A = \frac{\mathbb{C}[x_0, \dots, x_3]}{(F, G, H)}.$$

By hypothesis, for the generic choice of F', G' and H' of degrees $d - a, d - b$ and $d - c$ in A , we know that the degree d part of

$$\frac{A}{(F', G', H')}$$

is not zero. Now, consider elements F'', G'' and H'' of degrees $d + 1 - a, d + 1 - b$ and $d + 1 - c$. Notice that

$$d - a \geq d - b \geq d - c > c + b + a - 3$$

and recall that $A_i \simeq A_j$ as \mathbb{C} vector spaces if i and j are $> 2c + b + a - 3$. Thus for a general linear form L we have

$$F'' = LF^*, G'' = LG^* \text{ and } H'' = LH^*$$

and the forms F^*, G^* and H^* have degrees $d - a, d - b$ and $d - c$. Hence we have a isomorphism

$$(F'', G'', H'')_{d+1} \simeq (F^*, G^*, H^*)_d$$

and this is enough to conclude that the degree $d + 1$ part of

$$\frac{A}{(F'', G'', H'')}$$

is not zero and the result follows. \square

Lemma 5.4. *If $c \geq 13$, then the generic degree $d \geq 2c + 15$ surface of \mathbb{P}^3 does not contain a $CI(5, 12, c)$.*

If $c \geq 10$, then the generic degree $d \geq 2c + 12$ surface of \mathbb{P}^3 does not contain a $CI(6, 8, c)$.

Proof. We begin with the study of $CI(5, 12, c)$. Let $c = 13 + x$, $d = 2c + a + b - 2 = 41 + 2x$ and consider the ring

$$W = \frac{R}{(F, G, H, H')}$$

where the forms F, G, H and H' have degrees 5, 12, $13 + x$ and $28 + x$. The generic degree d surface does not contain a $CI(5, 12, c)$ if $H(W, 5) + H(W, 12) - H(W, d) < 0$, where $H(W, d) = H(W, 41 + 2x) = H(W, 14)$. Now we compute

$$H(5) = \binom{8}{3} - 1 = 55,$$

$$H(12) = \binom{15}{3} - \binom{10}{3} - 1 = 334,$$

$$H(14) = \begin{cases} \binom{17}{3} - \binom{12}{3} - \binom{5}{3} = 450 & \text{if } x > 1 \\ 449 & \text{if } x = 1 \\ 446 & \text{if } x = 0 \end{cases}.$$

Hence, $H(W, 5) + H(W, 12) - H(W, d) < 0$, and by Proposition 5.3 we conclude that the generic degree d' surface does not contain a $CI(5, 12, c)$ for $d' \geq d = 41 + 2x = 15 + 2c$.

The case of $CI(6, 8, c)$ is solved by completely analogous computations. \square

6. THE CASE $a > 4$: EXISTENCE RESULTS

Theorem 5.1 does not cover small values of a and b . In this Section we derive a result analogous to Theorem 4.2 in these cases.

We begin with proving two technical facts.

Proposition 6.1. *Let $a \leq b \leq c \leq d$ and $d \geq a + b + c - 3$. If a $CI(a, b, c)$ exists on the generic degree d surface, then it also exists on the generic surface of degree $d' > d$.*

Proof. Let $d' = d + 1$ and notice that it is enough to treat this case. The hypothesis reads as follows: the degree d part of the ring

$$\frac{A}{(F', G', H')}$$

is zero for generic forms F', G' and H' of degrees $d - a, d - b$ and $d - c$ where $A = R/(F, G, H)$. If $L \in A$ is a generic linear form, by [Sta80], we know that multiplication by L is an isomorphism in degree bigger than or equal to $a + b + c - 4$. Hence, the degree $d + 1$ piece of

$$\frac{A}{(LF', LG', LH')}$$

is zero and this is enough to complete the proof since, if three special forms, namely LF', LG' and LH' , have maximal span then the same property holds for a generic choice. \square

Lemma 6.2. *Let a, b and d be non-negative integers such that $4 < a \leq b$ and $d = 2a + 2b - 6$. If the generic degree d surface in \mathbb{P}^3 contains a $CI(a, b, a + b - 3)$, then the generic degree d' surface contains a $CI(a, b, c)$ for any $d' \geq a + b + c - 3$ and any $c \geq a + b - 3$.*

Proof. Notice that, by Proposition 6.1, the generic degree $d + \epsilon$ surface in \mathbb{P}^3 contains a $CI(a, b, a + b - 3)$ for all $\epsilon \geq 0$. Hence, by Lemma 3.2, the same holds for $CI(a, b, a + b - 3 + \epsilon)$ and surfaces of degree

$d + \epsilon$. Making ϵ vary and again applying Proposition 6.1 the result follows. \square

Theorem 6.3. *Let a, b, c and d be non-negative integers such that $a \leq b \leq c < d$. If $a = 5$ and $b \leq 11$, or $a = 6$ and $b \leq 7$, then a $CI(a, b, c)$ exists on the generic degree d surface of \mathbb{P}^3 . If $a = 5, b = 12$ and $c = 12$, or $a = 6, b = 8$ and $c = 8, 9$, then a $CI(a, b, c)$ exists on the generic degree d surface of \mathbb{P}^3 .*

Proof. To prove the thesis we combine all the previous results and technical facts. Crucial ingredients are also some explicit computations that we performed using the computer algebra system CoCoA [CoC04].

To determine whether a $CI(a, b, c)$ exists on the generic surface of degree d in \mathbb{P}^3 , we proceed as follows:

- if $c \leq a + b - 3$, we make explicit computations for all $d \leq a + b + c - 3$; a positive answer for $d = a + b + c - 3$ solves the cases for bigger d 's by Proposition 6.1;
- if $c = a + b - 3$ and $d \geq 2a + 2b - 6$, we verify each statement with an explicit computation for $d = 2a + 2b - 6$; if the answer is positive we conclude the same for $c \geq a + b - 3$ and $d \geq a + b + c - 3$ by Lemma 6.2.

We sketch this procedure for $a = 6$, the case $a = 5$ is completely analogous but lengthier. We need to perform explicit computations in the following cases:

- $CI(6, 6, c_1)$, for $c_1 \leq 9$ and $d_1 \leq 9 + c_1$;
- $CI(6, 7, c_2)$, for $c_2 \leq 10$ and $d_3 \leq 10 + c_2$;

The computations (see Example 6.4) show that the complete intersections exist on the generic surfaces of the required degrees. Hence we conclude that the generic surface of degree d contains a $CI(6, b, c)$ for all $b \leq 7$ and any c, d such that $d > c$. We conclude the proof for $a = 6$ by verifying existence in the cases: $CI(6, 8, 8)$ for $d = 19$, and $CI(6, 8, 9)$ for $d = 20$. \square

Example 6.4. We begin with verifying that the generic surface of degree $7 \leq d \leq 15$ contains a $CI(6, 6, 6)$. Using Proposition 3.1 it is enough to show that the ring

$$S = \frac{\mathbb{C}[x_0, \dots, x_3]}{(F, G, H, H', G', F')}$$

is zero in degree d , where the forms F, G, H, H', G' and F' are generic and have degrees $6, 6, 6, d - 6, d - 6$ and $d - 6$. Hence, for each d , we choose random forms with rational coefficients of the required degrees.

Then we ask CoCoA [CoC04] to compute the Hilbert function of S in degree d . Since for all d 's we get $H(S, d) = 0$, we conclude (by semicontinuity) that this is the case for a generic choice of forms of the appropriate degrees. In particular, as $15 = 6 + 6 + 6 - 3$ and $H(S, 15) = 0$, Proposition 6.1 yields that a $CI(6, 6, 6)$ exists on the generic degree $d \geq 15$ surface of \mathbb{P}^3 .

The same argument works in complete analogy for $c \leq 8$. For $c = 9$ we make an explicit computation for $d = 18$ and using Lemma 6.2 we show existence of a $CI(6, 6, c)$ on the generic degree d surface for $c \geq 9$ and $d \geq c + 9$. The cases for $c < d < c + 9$ are solved using Lemma 3.1 and the results for $c \leq 8$ and $a \leq 4$.

7. MAIN THEOREM AND FINAL REMARKS

We can now prove our main theorem:

Proof of Theorem 1.2. The existence part for the case $a \leq 4$ is Theorem 4.2 while existence for the remaining cases is Theorem 6.3. The asymptotic non-existences are given by Lemma 5.4 and Theorem 5.1. \square

Theorem 1.2 produces a complete asymptotic answer to our original question. We also get many existence and non existence results for small value of d . However, there are still infinitely many cases which we have not solved, e.g. $a = 7$ any b, c and d such that $7 \leq b \leq c$ and $c + 5 \leq d \leq a + b + c - 4$.

We state a conjecture completing Theorem 1.2:

Conjecture: *given non-negative integers a, b, c and d such that $a \leq b \leq c < d$, there exists a function $d(a, b, c)$, possibly assuming the value $+\infty$, such that the generic degree d surface in \mathbb{P}^3 contains a $CI(a, b, c)$ if and only if $d < d(a, b, c)$.*

As support for this conjecture, notice that it fits with the asymptotic statement and with the other results of Theorem 1.2. For example, $d(a, b, c) = +\infty$ for $a \leq 4$ and $d(a, b, c) < a + b + c - 3$ for $7 \leq a$.

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